PAPER MODELS OF ARCHITECTURAL SURFACES

IMAGES FOR IMPLICIT AND EXPLICIT GEOMETRIES

Caterina Cumino[1], Martino Pavignano[2], Ursula Zich[2]
[1] Politecnico di Torino
Department of Mathematical Sciences “Giuseppe Luigi Lagrange”
[2] Politecnico di Torino
Department of Architecture and Design
martino.pavignano@polito.it

ESSAY 32/02

GEOMETRY
PAPER MODELS
ARCHITECTURE
MATHEMATICS
INTERDISCIPLINARY KNOWLEDGE SHARING

This contribution focuses on paper models of architectural surfaces, in particular on some roofing systems describable by developable ones. Drawing on an interdisciplinary approach, between Architecture and Mathematics, potentialities and criticalities of these models in explicitly conveying Geometry are investigated, in relation to educational and communicative tasks, both when they are used in a direct, tangible way, and when the use is mediated by images generated by them (thus indirect); we discuss on the possibility for models and images to communicate their explicit and implicit Geometries.

The main issue discussed is that a material or analytical description unequivocally allows to grasp all the peculiarities of geometrical shapes, while other representations are subject to critical selection of data and are therefore affected by subjective interpretations; similarly, the translation of the physical model into images is the result of choices which emphasize certain object peculiarities over others and is thus less objective. Hence the importance of the physical model (as well as its digital counterparts) which, even if not used directly, can be complementary to a content that, alone, would be partial and/or misleading.

INTRODUCTION

The research described in this paper is part of a wider program aimed at investigating the relationships between Architecture and Mathematics by mean of Geometry, here intended as: a common language, a sum of methodologies and tools to foster architectural and mathematical education at university level, a theoretical but shared declination of Visual Thinking by using physical models of architectural elements. Our contribution is linked to studies on physical and tangible models of developable surfaces. We focus on the multiple communicative values of both images generated by paper models and images generating them; on issues related to the direct and indirect use of models and to their effective communication properties and features also through synthetic representation, namely through a critically selected set of images, in order to construct a visual narration that can highlight their meaning in relation to educational and communicative tasks. Moreover, we propose the use of physical models to foster and enhance the comprehension of architectural shapes, ranging from the study of their geometries to analytic description.

ARCHITECTURE, MATHEMATICS AND PHYSICAL MODELS: CULTURAL AND SCIENTIFIC BACKGROUND

The following paragraphs trace the state of the art for the recognized and consolidated use of physical models in Architecture and Mathematics. Strong historical bases highlight their importance as haptic declination of Visual Thinking, in the sense of Arnheim (1969) and Giaquinto (2011), not only as simple physical scaled reproduction of complex artefacts, but also as specialised instruments to investigate theoretical statements (Elser & Cachola, 2012; Friedman, 2018).

ARCHITECTURE AND PHYSICAL MODELS

In architectural practice the use of physical models draws its origins from the material representations, thus creation of tangible artefacts, used for religious ceremonies, for magic rituals, but also as scaled reductions/reproductions of everyday life scenes. Nonetheless, within these practices many scholars highlighted the celebrative, votive and ludic functions that humans charged models within the ancient times (Smith, 2004, pp. 3-17; Scolari, 2005, pp. 131-132; Barlozzini, 2013, pp. 45-49). One of the very first documented 'technical' interaction between Architecture and physical models can be found either as outcome of the process or example to be reproduced, between *neokóros* and *paràdeigma* (Scolari, 2005, pp. 131-132). Nowadays, at least in the architectural panorama, the general statute of the term model, however, is very complex and not only attributable to the definition of a data to be reproduced or copied (Ugo, 2008, p. 21), nor to a real or digital artefact. The physical model is itself the main result of a complex process of critical analysis: it is the synthesis of the architectural project or of the built space. In this case the model acts as haptic medium, while still presenting itself as an eidetic result of the previously cited processes. In this case, regardless of the specific function of the architectural model, be it votive, celebratory, design one, the model improved its role of simple physical representation, directly explorable by visual means (Bianchini 2007; Ribichini 2007, p. 50) or, on certain occasions, tactile when not immersive *ante litteram* (Docci, 2007, p. 25).

Physical models also played an important role in field which sees the intersection between architectural and structural design (Collins, 1963; Smith, 2004, pp. 89-124; Schilling, 2018, p. 25). Still, it is plenty of examples that highlight the model itself as a true self-contained project of architecture, or final expression of a process of critical/creative mediation between *archè* and *téchne*, in the strict sense (Rizzi, Piscitella & Rossetto, 2014, pp. 31-44). The contemporary debate sees the virtual model as undisputed protagonist for strengthening the intersection between the developments of "design as a kind of communication" and the strong links between "spatial intuition and image that concretizes it" (Albisinni, 2011, p. 71); while it is also been recognized as a complex meta-system of information (Brusaporci, 2019). Nonetheless, the physical, tangible, haptic architectural model still has a foundational role on several levels. Also, an important issue deals with the use of paper-based physical models, due to their low cost and specific dynamical characteristics, which qualify them to be similar to the family of three-dimensional origami models (Cumino, Pavignano, Spreafico & Zich, 2018a), with special regard to developable surface. There, we have already framed the role of origami inspired models as tangible extension of Visual Thinking both for Architecture and Mathematics (Cumino, Pavignano, Spreafico & Zich, 2018b).

MATHS AND PHYSICAL MODELS

In the history of Mathematics, the appearance of material mathematical models and their production/use, can be traced back to the second half of the nineteenth century, with the flourishing of Descriptive Geometry, up to the first decades of the twentieth century, when the prevalence of a more abstract point of view in mathematical research diminished their interest (Giacardi, 2015; Friedman, 2018).

In the nineteenth century, indeed, many mathematicians from universities and polytechnics across Europe, especially in Germany, devoted themselves to the manufacturing of material models: 'concrete' objects, made of plaster, string, wood, paper or cardboard, representing 3D geometric entities starting from their equations, had interactions both with research (to provide an effective mental image of the abstract objects of investigation) and with teaching at university level, not only in the mathematical field, but also in other disciplines such as Civil Engineering and Architecture.

Felix Klein (1872), one of the promoter of models production, expresses their meaning as a tool “to grasp the spatial figures in their full figurative reality, and (which is the mathematical side) to understand the relations valid for them as evident consequences of the principles of spatial intuition [*Anschauung*]” (Friedman, 2018, p. 123). Also, some models had moveable parts: from an educational point of view, this emphasized visual and haptic aspects of teaching mathematics, which were recognized and taken up again in the last century by Italian scholars about the so-called Intuitive Geometry (Castelnuovo, 1957, p. 91).

In summary, such models were meant to create a further, alternative way to represent mathematical entities. In fact, mathematical thinking is forced by its nature to use representations (Duval, 1999): set of symbols, formulas or visualizations through images external to the mind (such as diagrams, drawings, physical and virtual models, etc.) or visualizations through mental images; and the development of various registers of representation follows progress of Mathematics. As for visual representations, the set of mental processes related to the production and interpretation of images is a fundamental aspect in mathematical activity (Giaquinto, 2011) especially for Geometry.

Since here we focus on paper models of geometrically defined surfaces, we need above all to highlight geometric properties that characterize those surfaces which are representable by them. As it is known, developable surfaces are characterized by the possibility of being unrolled (developed) on a plane without stretching or tearing, namely without changing the measurements of angles and lengths. This is the reason why, in the present investigation, we only deal with developable surfaces. Mathematically this property is expressed by saying that a developable surface can be mapped isometrically on a plane; or, using the concept of curvature, it can be said that developable surfaces are characterized by an intrinsic (or Gaussian) zero curvature. These surfaces belong to the larger class of ruled ones, whose name (*surfaces réglées*) is due to the French mathematician Jean Nicolas Pierre Hachette (1769-1834) and it means that one can always find at least one way to put a ruler (i.e. a straight line) on them.

A ruled surface is generated by the movement of a straight line in the space: it is enough to assign a director curve, parametrically identified by the point $Q(u)$, where u varies in an interval contained in the real line; so, the surface is represented by an expression of the form $S(u, v) = Q(u) + v\,r(u)$ where for each $u = u_0$, $S(u_0, v) = Q(u_0) + v\,r(u_0)$ describes a line (generator) passing through $Q(u_0)$ and having direction of the vector $r(u_0)$. Developable surfaces are a particular type of ruled surfaces: they were independently studied around the end of the eighteenth century by the mathematicians Leonhard Euler (1707-1783) and Gaspard Monge (1746-1818), using differential calculus and investigating the ways of constructing them.

Monge used thread models of developable surfaces in his lessons at the *École Polytechnique* in Paris. One can prove that a ruled surface is developable if the tangent plane to the surface, at each point of a generator line, is constant and if this property is verified for each generator line. Otherwise the ruled surface is said to be non-developable. Moreover, the generator lines of a developable surface may pass through a fixed point (at infinity or not): this characterizes the cylindrical and conical surfaces respectively; alternatively, the generator lines may be tangent to a given space curve and in such a case the developable surface is called the tangent developable to the space curve.

METHODOLOGICAL APPROACHES TO PAPER MODELS: EXPLICIT VS IMPLICIT GEOMETRIES

In the following paragraphs we will use both the adjectives 'explicit' and 'implicit' with respect to Geometry, but before entering the issues related to such discipline, we remark that we are referring to such terms with the conventional definition we can find on dictionaries: *Treccani* for Italian original terms, *Oxford* for the corresponding English ones. By using the word 'explicit' we then refer to what is or can be clearly expressed, without any hints of misconception; otherwise with the word 'implicit' we mean a concept that, even without being formally and/or expressly stated, is necessarily involved somewhere else.

PAPER MODELS FROM THE SCHILLING'S *CATALOG*, 1911

It is clear that every physical model (as a designed and accomplished artefact) has an implicit geometry, which allowed its execution and an explicit geometry, perceived at the time of its direct use, whether tangible or visual, or of its indirect use, mediated by the production of images. With regard to the relationships between implicit/explicit and direct/indirect, we refer to a series of paper models described, by means of text and image, within catalogs of mathematical models published mainly in Germany between the late nineteenth and early twentieth centuries. In this context, the physical models of algebraic surfaces produced by the mathematicians Felix Klein and Alexander von Brill were so successful to trigger series production and sales to a specialized broad public (Giacardi, 2003). The sets of models were the subject of a series of punctual publications, subsequently were organized by Martin Schilling in the form of a double-key *Catalog*, one chronological and one thematic, starting from 1903: *Catalog mathematischer Modelle für den höheren mathematischen Unterricht* (Schilling, 1903). The *Catalog* represented, in its various editions, an opportunity for systematizing tangible models with an explicit teaching value (Neuwirth, 2014). Among the many models presented, handcrafted with poor materials (Fischer, 2017), the paper ones offer a variety of shapes obtained through different abstraction modalities, through the choice of characterizing and descriptive elements of the surface in question, and realization processes.

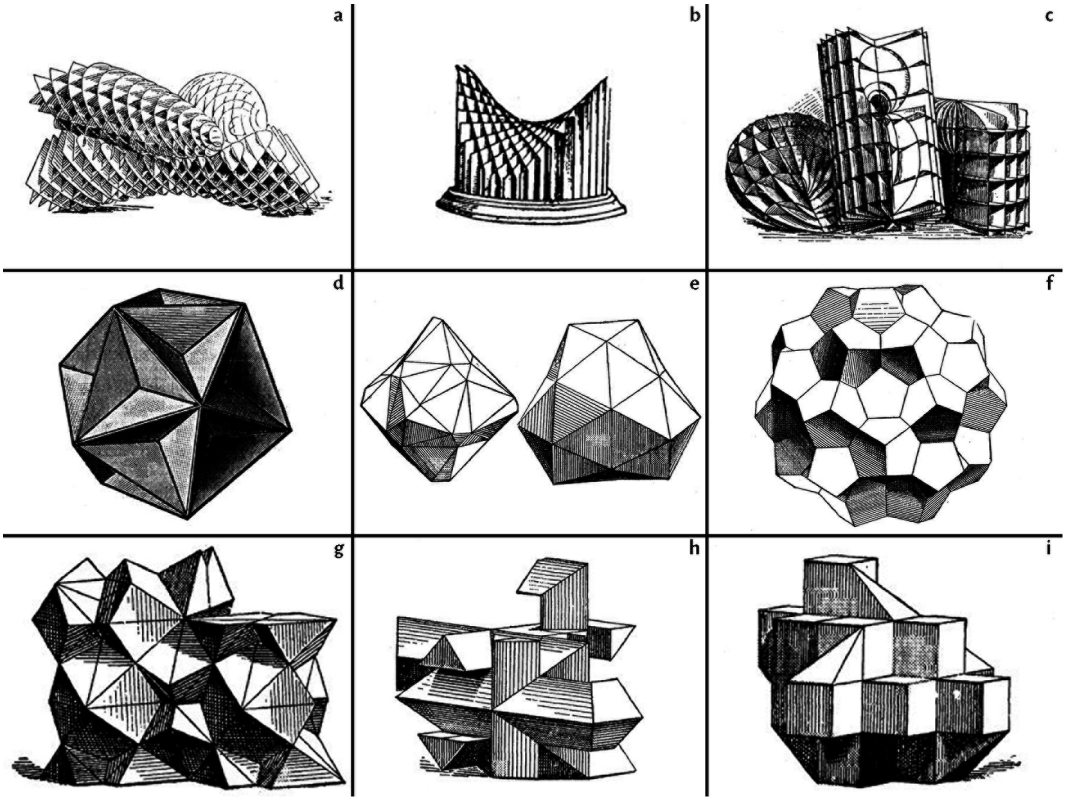


Fig. 1 Models from Schilling (1911): a) Cart.-S., 1, p. 111; b) Cart.-S., 6, p. 114; c) XXII, 1-3, p. 136; d) XXXVII, 3, p. 149; e), f) XV,8-12, p. 156; g), h), i) XIX,1-12, p. 170.

Figure 1 summarizes all the images of paper models in the *Catalog* published by Schilling (1911) (9 out of 99 overall images, two of which in the first part organized by series and 97 in the second thematic part, even though the models described in the *Catalog* are many more then the portrayed 99). Given their eidetic value, the 9 images chosen were considered, on the basis of their communicative values, sufficiently exhaustive to exemplify the relative surfaces. One can recognize models made as a sequence of section planes, where the surface geometry is represented through a completion of meaning (Figures 1a, 1b, 1c); among them one can observe a movable model (Figure 1a) and a fixed one (Figure 1b); a model is faceted by planes (Figure 1d), the other ones are identified by a series of vertices (Figures 1e-1i). Comparing with the photographic images (e.g. Figure 1c with Figure 2) of physical models in the *Tübingen Collection*,

Fig. 2 Cardboard models from the collection of the *Museum der Universität Tübingen, Brill's Catalog* (1895): a) XXII,1; b) XXII,2; c) XXII,3 (Seidl et al. 2018, pp. 306, 308).

it can be observed how similar they are to their synthetic and cumulative representation of the *Catalog*: the 'live drawing' description of surfaces is also highlighted by the graphic dressing of shadows underlining the structure depth of planes intersection, so describing by 'absence' their tangent surface.

Overall, one can remark that only a few models are represented in relation to their support plane and that the shadows are not geometrically structured: for example, parts of the plane are dark filled without being shaded by other surfaces (Figure 1f). On the contrary, by comparison with other representations of similar surface models in previous and subsequent catalogues or journals, one can observe that: except Figure 1d, all of them are already present in the Schilling's *Catalog* of 1903; models created by section planes had already been published, partly with different graphic peculiarities (compare Figure 1a and Figure 3a), partly in a similar way (Figures 1c, 3c); all models are confirmed from edition to edition except one (Figure 3b), present in the Walter Dyck's *Catalog* of 1892 as part of a group of four models (Figures 1e, 1f): this is the only one which has no longer been reproduced graphically (the others were distributed punctually in the columns text of specific descriptions).

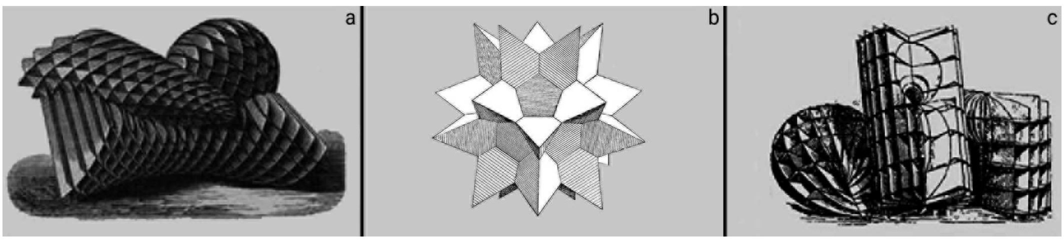


Fig. 3 Pictures of cardboard models from different sources: a) Detail of the publisher L. Brill's Prospectus, 1884 (Seidl et al., 2018, p. 309); b) Model of projection from the fourth dimension (Dyck, 1892, p. 254); c) Detail of an advertisement by L. Brill, 1895 (Seidl et al., 2018, p. 312).

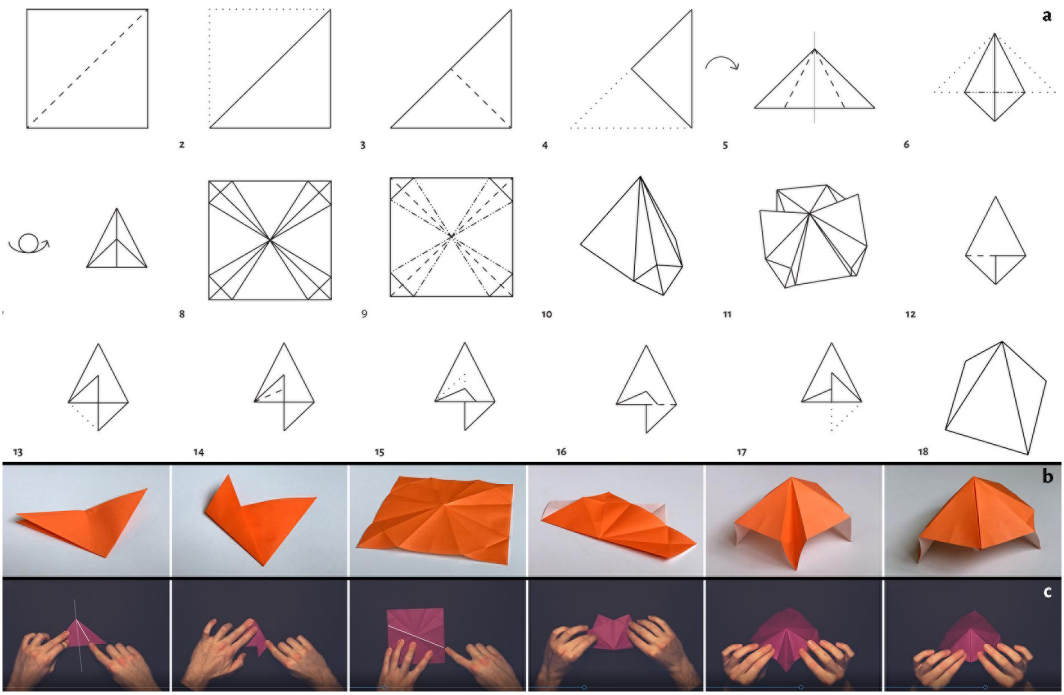
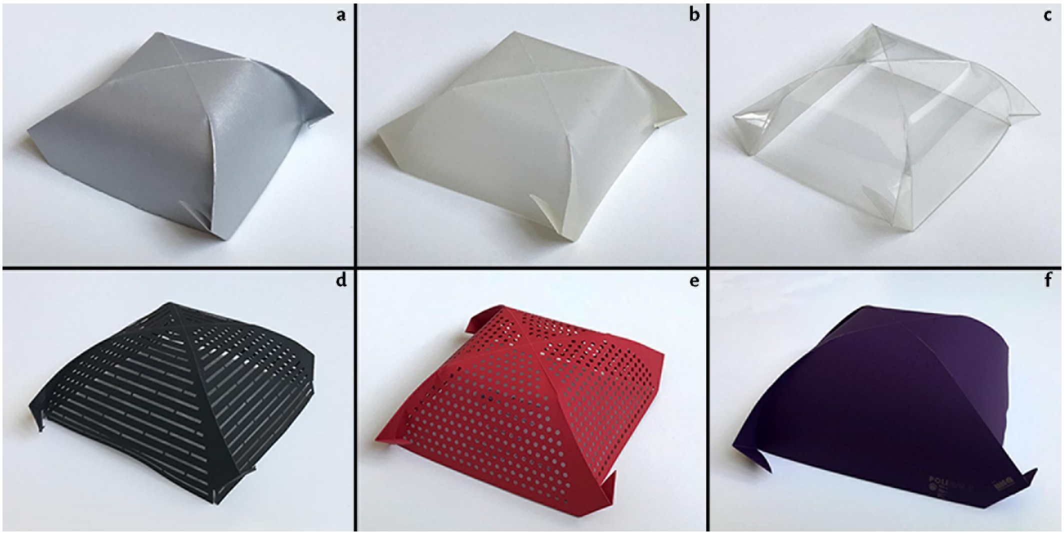


Fig. 4 Square based pyramid: folding sequence through different languages: a) folding sequence; b) photographic shots from a physical folding sequence; c) frames from a folding video tutorial.

ORIGAMI INSPIRED MODELS OF SOME ROOFING SYSTEMS: RESULTS BETWEEN SIGNIFIER AND MEANING

The critical analysis of previously presented models, through their graphic and textual description, offered a basis for an interdisciplinary approach to the systematization of geometric peculiarities for each paper model we are going to propose for the communication of roofing systems geometries. Seeing and perceiving geometries of the built shapes are the result of a mediation between theoretical approach and abstraction skills, in order to recognize concrete geometries. In this sense, our paper models of developable surfaces recognizable in pitched roofs and vaulted ceiling can be considered representative of the architecture in question even if not respecting its material and structure: they are an expression of synthesis in description of the built, without thickness and/or rigor, and they favour accessibility to constituent/latent geometries, not always immediately recognizable in real dimension of the built, being models for communication and sharing of theoretical surfaces geometric properties. With this intention, paper models simultaneously assume the role of medium for direct exploration and for generation of images that allow various points of view, thus becoming effective communicators, as expressions of Intuitive Geometry. Among the possible paper models, those similar to origami must be considered dynamic geometries even if the final product is static (Lang, 2018): they are transformation of paper sheets, thus the succession of folds shaping them is part of their nature (Friedman, 2018). Each folding step is a geometry implicit in the model, expression of its construction process, a condition without which the final product would not have the designed shape. Meanwhile, the process outcome does not always explicitly show this geometry, where these models as image generators already during the modeling process. Observing roofing systems, one can compare various descriptions of the same folding sequences for a model defined by intersection of planes. Figure 4 summarizes some expressive modalities, essentially graphic without text support, for the transformation of a two-dimensional sheet into a three-dimensional model. Each modality has been tested with a heterogeneous audience and all have proved effective in

Fig. 5 Cloister vault: origami inspired paper models made by different material with different techniques, highlighting geometric proprieties: a) die cut silver paper 200g; b) die cut tracing paper 90g; c) die cut light acetate; d) laser engraved black paper 220g with holes to let the user see through the model and to show straight lines on the semi-circular ruled surfaces; e) laser engraved red paper 150g with holes to let the user see through the model; f) laser engraved purple paper 220g with partial abrasion of material to optimize the model closure.



guiding the modeling of the pyramid, despite the fact that they have actually intercepted different users' targets, highlighting the medium/user ratio specific to each language (Cumino, Pavignano, Spreafico & Zich, 2017). In Figure 4a, the folding sequence is complete, it uses the origami language both in the type of lines and in the symbology (Lang, 2011, pp. 11-40). In Figure 4b there are only some of the possible photographic shots during the folding phase as well as in Figure 4c some frames are displayed from the illustrative video of the folding sequence: photos and videos have proved to be more shareable tools, as they are less specialized then less pretentious in terms of basic preparation. The fold sequences create a series of images that become an expression of implicit geometries suitable also for indirect use: they illustrate geometric peculiarities that are not necessarily recognizable in the finished model. Nonetheless, images taken from the video sequence partially retain dynamic and physical qualities of 'keeping in memory' the folding steps necessary to achieve the final form (Demaine, Demaine, Hart, Price & Tachi, 2009; Akitaya, Mitani, Kanamori & Fukui, 2015; Lang, 2018). The video sequence, optimized by digital elaborations, is perfectly suited to share and disseminate the geometric dimension of the origami modeling process. In all cases, the origami model can therefore be used both to express and to communicate abstract concepts, provided one checks related terminology, as well as visual results.

A different role is played by Crease Pattern (from now CP): the set of conventionally shared signs that illustrate the folds to be carried out, optimized for production, without showing the sequence. Table 1, lines 1, 6, show the pyramid CP by recognizing the implicit geometry of the model without having the possibility of prefiguring the finished product. By comparing the fold sequence (Figure 4a, step 8) with the CP (Table 1, A1, A6) one can observe the small differences related to the geometries that step by step the sequence carries behind leaving traces on the paper which are not necessary for modeling, thus superfluous in a CP optimization phase. Therefore, starting from a fold sequence, or reopening a model, we always find the CP lines and not only, because many other lines are created in the various passages producing a summation of folds that are difficult to interpret. Starting from a CP, on the other hand, it is not necessarily possible to reconstruct the folding sequence that defined it, since it is not always the result of a geometric chain and, especially in the management of curved surfaces, it could be a consequence of surfaces development and overlapping that cannot be reconstructed step by step. Hence, CP definition allows the origami design to overcome the limits of folding sequence sharing and repeatability, shifting the problem core on the choice of materials and instrumentation: it can be drawn (directly or indirectly) or reproduced by pressure or engraved by means of a laser ray. It is therefore clear that the way in which its design is traced constrains the choice of materials (Figure 5).

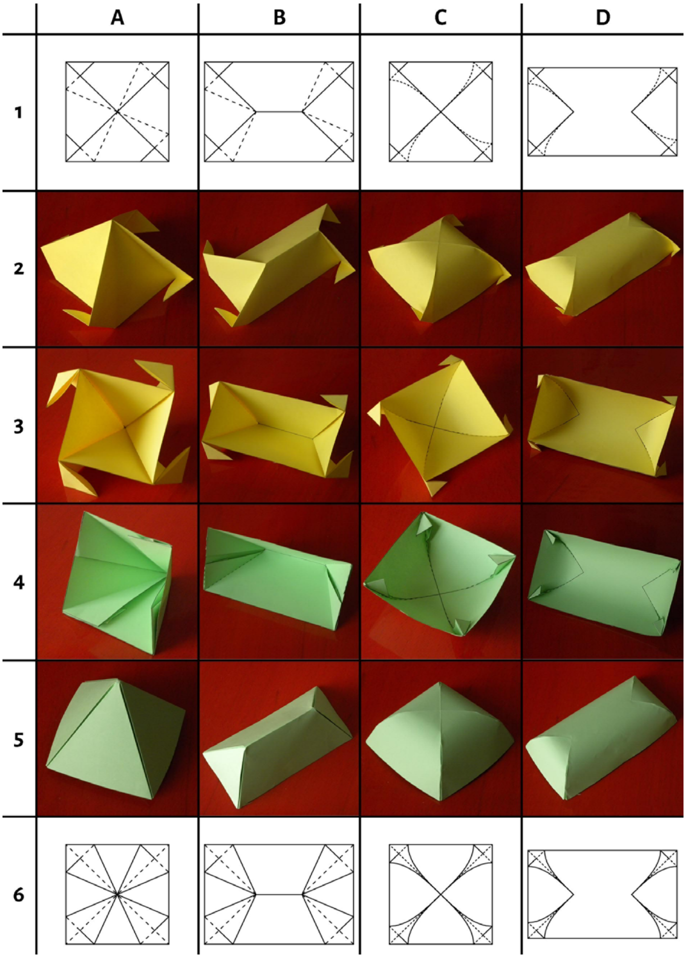
DISCUSSION ABOUT DIRECT AND INDIRECT FRUITION OF THE PAPER MODEL

In an education/dissemination context, it is necessary to distinguish between seeing the model and observing its geometric features, between the practice of communicating some geometric ideas through the model and communicating the model and its peculiarities. Models direct use is a shared issue (Megahed, 2017; Hemmerling & De Falco, 2018), considering them as a media for education/dissemination about those architectural artefacts belonging to the large family of Cultural Heritage (Solima, 2012). Their indirect management must be critically evaluated: in absence of the tactile or direct visual exploration relationship, it appears to be a still little explored territory, considering them as models to represent themselves as images to be shared. In this sense, "images [...] bring to light a set of elements and dynamics that engage the observer from a cognitive point of view in multiple ways. Furthermore, the points of view from which we observe them must be manifold" (Luigini, 2017, p. 2). These remarks led us to deduce methods to use of images derived from the models we designed. Firstly, if images have to communicate the model as such, they have to represent the model in its volumetric and formal complexity, highlighting its material characteristics. Secondly, if images have to communicate particular aspects of the model, they have to show its constitutive geometries. In summary, the increasing complexity of visual communication involves the generation of multipurpose images. Moreover, if the model is the result of an interdisciplinary comparison process, the image derived from it becomes the visual outcome of a design process, graphic as well, for the communication of the different 'model souls' , because each artefact is a set of images disclosed by objects, but also (visual) expression of its iconic meaning through images (Gay, 2015, pp. 169-171). As a consequence of our present outcomes, we propose a small integration in the statute of the term model, giving it the role of image generator, according that the origami/paper model is the result of a design process, synthesis between Representation and Geometry.

DIRECT FRUITION BETWEEN VISUAL AND HAPTIC EXPLORATION

In modeling vaulted surfaces, the description of extrados and intrados does not necessarily coincide with the concept of *recto* and *verso*, since it should be functional to explain either one or the other surface without the presence of the extra-paper, for the communicative purposes of the model. Therefore, it is possible to design different solutions to describe and fold the same architecture (Cumino, Pavignano, Spreafico & Zich, 2018a, p. 87)

Table 1, lines 1, 6 show that the same shape has different representations/CP according to the management of the extra-paper. With extra-paper outside is indifferent to deal with cylinders having as a cross section a broken straight line or a curved one; for extra-paper inside in the first case it is possible to bring it closer to the translation surface minimizing its perceptual impact, while in the second one this is not possible and therefore the extra-paper inside disturbs the reading of intersection curves without allowing an optimal reading even from the outside, because segmentation points of the



Tab. 1 CP and origami models of covering surfaces: A) Pyramid; B) Pitch roof; C) Cloister vault; D) Barrel vault with cloister heads; 1) CP with extra-paper outside; 2) models extrados with extra-paper outside; 3) models intrados with extra-paper outside; 4) models intrados with extra-paper inside; 5) models extrados with extra-paper inside; 6) CP with extra-paper inside. 1C is the result of a research described in Cumino et. al (2015).

intersection curve arise in correspondence to inside paper stratifications. The outside extra-paper solution respects the geometric rigor of the project, while the inside one is rigorous only for planar surfaces. This model, as a support to the visit in person of the described spaces, privileges the description of accessible areas to create a direct, visible and tangible connection with the artefact and subsequently allows integration of other information not directly visible in the built (Armand, Cumino, Pavignano, Spreafico & Zich, 2018).

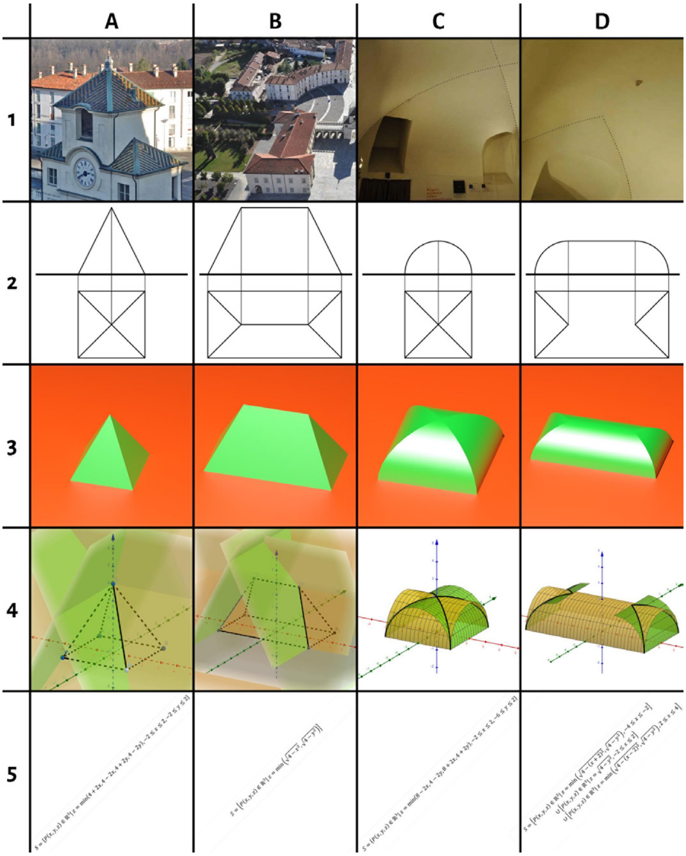
In this case, the model must have finishes that do not interfere with geometric perception of the visible.

INDIRECT FRUITION BETWEEN DESIGN AND COMMUNICATION

Problems related to direct exploration of a paper model lead to evaluate its design in close relation with its communicative intention. Table 2 shows relationships between real object (row 1), theoretical surface recognized as the description of the built, representation on projection planes (row 2), 3D modeling (row 3), *GeoGebra* modeling (row 4) and analytical description (row 5). The critical observation of the built object is exemplified by facing four different problems. Roofs in columns A, B are not entirely perceptible in a direct way: their vision is limited by the height of the point of view; these surfaces are defined by intersections of cylinders with a broken straight line as cross section, highlighted by the arrangement or by the choice of materials. On the other hand, vaults in columns C, D, are intersections of cylinders with a curvilinear cross section and the corresponding surfaces are represented precisely from the user's point of view; here intersections are not highlighted by structural or decorative elements. The proposed surfaces are those closest to the real situation: it is rare to observe a pitched roof from its intrados and the consistencies of a vaulted surface from its extrados (usually this one is covered and inaccessible or even embedded in, for example, the upstairs floor), thus overcoming any limits related to the actual accessibility of the point of view.

Similarly, the use of orthographic projections, with projection centre at infinity and therefore far from perception of reality, has the same purpose of representing such surfaces in the most objective and synthetic way, to describe their geometry. The limit of this representation is its 'specialization' governed by a very precise and unintuitive coded language.

Rows 2, 3 allow a comparison of representations on projection planes and 3D modeling, through the choice of a similar point of view (from the outside). The result is an immediacy of the three-dimensional speech with respect to that of the projection planes and the non-exhaustiveness of both presentations, with the added value of the interaction



Tab. 2 Representation of covering surfaces: A) Pyramid; B) Pitch roof; C) Cloister vault; D) Barrel vault with cloister heads; 1) Real objects pictures; 2) Orthographic projections of their theoretical surfaces; 3) CAAD 3D modelling of theoretical surfaces; 4) *GeoGebra* 3D modelling of theoretical surfaces; 5) analytic descriptions.

between them. Comparing the communicative potential of these representations, also origami in Table 1, we can underline how different the comparison with the haptic model is, rather than doing it with the images derived from it. As for rows 3 and 4, one should remember that, unlike other fields of knowledge, there exists no other ways of gaining access to the mathematical objects but to produce some semiotic representations of them; moreover, a mathematical processing always involves substituting some semiotic representation for another, namely the representation of an object is 'translated' into a different representation of the same object (Duval, 1999). The set of equations and inequations in row 4 analytically describes a locus of points in the space, while row 3 shows the corresponding visual translation easier to grasp for everybody, obtained by a 3D *Geogebra Calculator*. In a certain sense this path refers to the editorial choices of the aforementioned Schilling's *Catalog* (1911), where only two registers were used to present mathematical models: natural language and images, not equations. Moreover, *GeoGebra* models proposed here can be interpreted as image generators providing information on the geometric genesis of the study objects, complementing that of corresponding paper models; just consider, for example, the case of cloister heads vault in column D, where the *GeoGebra* model highlights the cylinders whose intersection generates the surface. However, the fruition of the 3D *GeoGebra* dynamic model, instead of a set of its static images, would be much more easy-to-understand and 'explorable' in a direct way, although not replaceable to the physical model, for the visual analysis of its implicit geometry. Graphic representations as above, i.e. static images generated by *GeoGebra*, raise some communication issues: Tab. 2 A3, B3 show that the geometric genesis of both the pyramid and the pitch roof requires quite a high abstraction capability, to be understood. The user has to recognize the shape faces following the continuous/dashed segments which mark the intersections of planes containing them; similarly, in C3, D3. *GeoGebra* does not provide a simple representation, as it does not remove that portions of cylinders not belonging to the described shape. Nonetheless, referring to Table 2, D3 one can see that the geometric generation of the architectural shape is more comprehensible than the C3 one, even if the *GeoGebra* represents the both in the same way.

EXPLICIT/IMPLICIT, DIRECT/INDIRECT

Consider for example the pyramid model (column A in Table 1 and Table 2); one can see that the geometry shown by the origami model is evidently linked to the defining plane surfaces; however, the relationship full/empty is unresolved (compare Table A2): the origami model, due to its design, has no basis and it is not explicit whether it is empty or not; moreover, the absence of a basis is explicit in its direct use while it is not equally explicit in the indirect one thought images. The non-exhaustiveness of geometric description produces a misunderstanding also in other representations; this highlights the effectiveness of direct exploration of tactile models to complement the description of a surface. An implicit geometry is added to the explicit geometry, unequivocally communicated by the model direct use, belonging to the model design path; this latter, in an origami model, can be enjoyed directly by reading the CP, or indirectly through its images. Direct fruition manipulating the CP further completes the communication by enriching the message with dynamism of transformation, at the same time expressing implicit geometry of final model and explicit one of the geometry of single fold. In the pyramid origami model we can therefore recognize as implicit geometry not only the one underlying the design process but also the one in the production process. This applies to each of the models presented here.

CONCLUSIONS

In this contribution we have drawn on the analysis of mathematical paper models and on design experiences of paper models of some roofing surfaces describable by developable ones; working with an interdisciplinary approach, between Architecture and Mathematics, we investigated some potentialities and criticalities of these models in explicitly conveying Geometry, both when they are used in a direct, tangible way, and when the use is indirect, mediated by images. As each language has its specificity, each image is a form of representation and "the main difficulties can be found in the transliteration of concepts and arguments from one discipline to another when they take on different meanings depending on the context –the term 'representation' alone is a clear example" (Luigini, 2019, p. 180).

In general, Geometry that can be communicated varies in relation to the type of use/users: when using physical models in deferred/indirect mode, or rather through interposed medium, it is important to pay attention to criticalities of their representation with respect to relationships between signifier and meaning.

It is a fact that a material description and an analytical one are unambiguous and allow unequivocally to grasp all the peculiarities of geometrical shapes, while other representations are subject to critical selection of data and are therefore affected by subjective interpretations. Similarly, the translation of the physical model into images is the result of choices emphasizing certain object peculiarities over others and is thus less objective. The relationship between message and receiver is indeed conditioned by context, background, language, experience and not only. A significant example is provided by Schilling's *Catalog* where textual description of geometrical surfaces supported by few symbolic images with even less analytical descriptions was considered sufficient for mathematicians (and indeed it was) to understand surfaces geometry in a unique way. Nowadays: more or less sophisticated computer-graphics programs allow to represent geometric objects starting from their analytical description, obtaining virtual models that can be observed/manipulated in a virtual space, to better understand their geometric properties, thus enriching the visual and intuitive component in the study of Geometry.

The importance of the physical model arises both from its physicality and from the transmission potential of the geometries related to it (Gay, 2000), since, even if not used directly, it can be complementary to a content that, alone, would be partial and/or misleading. Last but not least, physical models (as well as their digital counterparts) are both generated by and possible generators of a consistent and articulated set of images.

ACKNOWLEDGEMENTS

For the authors M. Pavignano and U. Zich the contribution was developed within the research project M.A.G.I.A. 2019: Mathematics Architecture Geometry. Interconnections and Applications, 2019.
C. Cumino is member of the National Group for Algebraic and Geometric Structures, and their applications (GNSAGA-INDAM).
Although the contribution was conceived jointly, paragraphs were authored in this way: Introduction; Architecture, Mathematics and physical models: cultural and scientific background; Origami inspired models of some roofing systems: results between signifier and meaning; Discussion about direct and indirect fruition of the paper model; Direct fruition between visual and haptic exploration; Indirect fruition between design and communication; Explicit/Implicit, Direct/Indirect, Conclusions by C. Cumino, M. Pavignano and U. Zich; Architec-

ture and Physical models, M. Pavignano; Maths and physical models C. Cumino; Methodological approaches to paper models: explicit vs implicit geometries; Paper models from the Schilling's *Catalog*, 1911, U. Zich.
Figures credits: Fig. 1 Schilling, (1911). Fig. 2 Museum der Universität Tübingen. Fig. 3a, 3c Museum der Universität Tübingen; 3b Dyck (1892). Fig. 4a Instructions were elaborated in 2016 by V. Bosetto during her trainee at Polito DISMA with the Centro Studi Residenze Reali Sabaude in Venaria Reale, Italy; Fig. 4b Authors; Fig. 4c frames were extracted from a video made in 2018 by P. Farina during his trainee at Polito DISMA with the Centro Studi Residenze Reali Sabaude in Venaria Reale, Italy. Fig. 5 Authors.
Tables credits: Tab. 1 Authors; pictures by A. Manino. Tab. 2 Authors; A1, C1, D1 U. Comollo; B1 Google Maps. We warmly thank the Centro Studi Residenze Reali Sabaude for granting us the permission to use pictures of the Royal Residence of Venaria Reale; Giovanni Berruto (Polito – DAD, ModLabArch) for his technical support in prototyping and producing our models; Ornella Bucolo and Daniela Miron (Polito – DAD, LabRilDoc) for their technical support in photographing our models.
Crease Patterns credits: Authors.
All our models were folded by C. Cumino.
All graphic elaborations were made by M. Pavignano.

DECLARATION OF INTEREST STATEMENT

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

REFERENCES

Akitaya, H.A., Mitani, J., Kanamori, Y., & Fukui, Y. (2015). Unfolding simple folds from crease patterns. In K. Miura, T. Kawasaki, T. Tachi, R. Uehara, R. J. Lang & P. Wang-Iverson (Eds.). *Origami 6, I: Mathematics of Origami* (pp. 85-96). Providence, RI: American Mathematical Society.

Albisinni, P. (2011). L'analisi grafica dell'architettura: dall'analogico al digitale. In: P. Albisinni & L. De Carlo (Eds.). *Architettura disegno modello. Verso un archivio digitale dell'opera di maestri del XX secolo* (pp. 65-76). Roma, IT: Gangemi.

Armand, P., Cumino, C., Pavignano, M., Spreafico, M. L., & Zich, U. (2018). Geometry and origami to share cultural heritage: results of the experimentation "The King and the Origami" at the Royal Residence of Venaria. In *Bridges Stockholm 2018. Mathematics, Art, Music, Architecture, Education, Culture. 2018 Conference Proceedings* (pp. 375-378). Phoenix, AZ: Tesseletion.

Arnheim, R. (1969). *Visual Thinking*. Berkeley, CA: University of California Press.

Barlozzini, P. (2013). *Il modello fisico in architettura. Uno strumento di rappresentazione tanto arcaico quanto attuale* (2nd ed.). Roma, IT: Aracne.

Bianchini, C. (2007). *Dal reale al virtuale (e ritorno): il modello ligneo di Antonio da Sangallo per il Nuovo San Pietro in Vaticano. Disegnare Idee Immagini*, 34, 36-49.

Brusaporci, S. (2019). The Visual Bride: representing tangible heritage between digitality and real contents. *IMG journal*, 1(1), 74-91.

Collins, G. R. (1963). *Antonio Gaudi: structure and form*. Perspecta, 8, 63-90.

Cumino, C., Frigerio, E., Gallina, S., Spreafico, M. L., & Zich, U. (2015). Modeling vaults in origami: a bridge between mathematics and architecture. In K. Miura, T. Kawasaki, T. Tachi, R. Uehara, R. J. Lang & P. Wang-Iverson (Eds.). *Origami 6, II: Technology, Art and Education* (571-582). Providence, RI: American Mathematical Society.

Cumino, C., Pavignano, M., Spreafico, M. L., & Zich, U. (2017). Disclose geometry to educate to shape reading. In *Edulearn 2017 Proceedings* (pp. 2035-2042). Valencia, ES: IATED.

Cumino, C., Pavignano, M., Spreafico, M. L., & Zich, U. (2018a). The Mole Antonelliana between real shape and folding design. Origami7. In *The Proceedings from the 7th International Meeting on Origami in Science, Mathematics and Education*. Volume One: Design, Education, History, and Science (pp. 73-88). St. Albans, UK: Tarquin.

Cumino, C., Pavignano, M., Spreafico, M. L., & Zich, U. (2018b). The Origami model as geometrical extension of Visual Thinking. In *Rappresentazione/ Materiale/Immateriale. Drawing as (in)tangible representation. 40th International Conference of Representation Disciplines Teachers* (pp. 1279-1286). Roma, IT: Gangemi.

Castelnuovo, E. (1957). L'*insegnamento della matematica nella scuola preelementare e elementare*. Scuola e città, 3, 93-98.

Demaine, E.D., Demaine, M. L., Hart, V., Price, G.N., & Tachi, T. (2009). *(Non)Existence of pleated folds: how paper folds between creases. Graphs and Combinatorics*, 27, 377-397.

Dyck, W. (1892). *Katalog matematischer und mathematisch-physikalischer Modelle*. Munchen, DE: Wolf & Sohn.

Docci, M. (2007). *La Basilica vaticana non costruita. L'opera di Antonio da Sangallo. Disegnare, idee immagini*, 34, 24-35.

Duval, R. (1999). Representation, vision and visualization: cognitive functions in mathematical thinking. basic issues for learning. In F. Hitt, & M. Santos (Eds). *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 3-26). Columbus, OH: Eric.

Elser, O., & Cachola Schmal, P. (Eds.). (2012). *Das Architekturmodell: Werkzeug Fetisch kleine Utopie*. Zürich, CH: Scheidegger Und Spiess Ag Verlag.

Fischer, G. (Ed.) (2017). *Mathematical models. from the collections of universities and museums.* Wiesbaden, DE: Springer Spektrum.

Friedman, M. (2018). *A history of folding in Mathematics. Mathematizing the margins*. Basel, CH: Birkhäuser.

Gay, F. (2000). *Intorno agli omolografi. Strumenti e modelli per la geometria descrittiva*. Venezia, IT: Istituto Universitario di Architettura di Venezia.
Gay, F. (2015). L'incontenibile concretezza dell'eidos: ideazione ed evoluzione degli artefatti. In P. Belardi et al. (Eds.) *Visualità. Idee per la Rappresentazione: atti del seminario di studi* (pp. 161-180). Roma, IT: Artegrafica PLS.
Giacardi, L. (2003). La collezione di modelli geometrici della Biblioteca speciale di Matematica "G. Peano". In G. Giacobini (Ed.), *La memoria della scienza*. Musei e collezioni dell'Università di Torino (pp. 251-266). Torino, IT: Alma Universitas Taurinensis.
Giacardi, L. (2015). *Geometric models in mathematics teaching in Italy at the turn of the twentieth century. Mathematisches Forschungsinstitut Oberwolfach*, 47, 2784-2787.
Giaquinto, M. D. (2011). *Visual thinking in mathematics: an epistemological study*. Oxford, UK: Oxford University Press.
Hemmerling, M., & De Falco, C. (2018). ArchiFold: an educational approach for the integration of mathematics in the architecture curriculum. In *Nexus Architecture and Mathematics 2018. Conference Book* (pp. 221-226). Torino, IT: Kim Williams Books.
Klein, F. (1872). *Vergleichende betrachtungen über neuere geometrische forschungen*. Erlangen, DE: Duchert.
Lang, R. J. (2011). *Origami design secrets: mathematical methods for an ancient art* (2nd ed.). Boca Ranton, FL: CRC Press.
Lang, R. J. (2018). *Twist, tiling and tessellations: mathematical methods for geometric origami*. Boca Ranton, FL: CRC Press.
Luigini, A. (2017). "Immagini?" as an Interdisciplinary Proposal. In *Proceedings of the International and Interdisciplinary Conference IMMAGINI? Image and Imagination between Representation, Communication, Education and Psychology*. Basel, CH: MDPI. Doi:10.3390/proceedings1091109.
Luigini, A. (2019). Notes about interdisciplinarity and transdisciplinarity of images. *IMG journal*, 1(1), 174-183.
Megahed, N. A. (2017). *Origami Folding and its Potential for Architecture Students. The Design Journal*, 20(2), 279-297.
Neuwirth, S. (2014). Les «objets mathématiques» comme modèles mathématiques: introduction historique et inventaire. In E. Guigon (Ed.), *Objets mathématiques*. Milano, IT: Silvana Editoriale.
Ribichini, L. (2007). *I Modelli. Storie di diverse vicende e realizzazioni. Disegnare Idee Immagini*, 34, 50-61.
Rizzi, R., Pisciella, R., & Rossetto, A. (2014). *Il Daimon di Architettura*. Manuale. Sesto San Giovanni, IT: Mimesis.
Schilling, M, (1903). *Catalog mathematischer Modelle für den höheren mathematischen Unterricht*. Halle, DE: Martin Shilling.
Schilling, M. (1911). *Catalog mathematischer Modelle für den höheren mathematischen Unterricht*. Leipzig, DE: Martin Shilling.
Schilling, A. (2018). *Architecture and model building: concepts, methods, materials*. Berlin, DE: De Gruyter.
Scolari, M. (2005). *Il disegno obliquo. Una storia dell'antiprospettiva*. Venezia, IT: Marsilio.

Seidl, E., Loose, F., & Bierende, E. (Eds.). (2018). *Mathematik mit Modellen. Alexander von Brill und die Tübinger Modellsammlung*. Tübingen, DE: Museum der Universität Tübingen MUT.

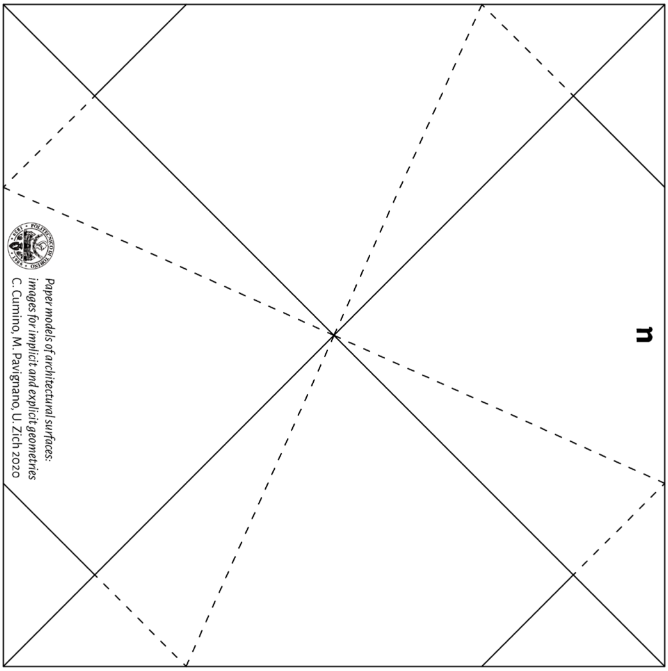
Smith, A. C. (2004). *Architectural Model as Machine. A New View of Models from Antiquity to the Present Day*. Oxford, UK: Elsevier-Architectural Press.

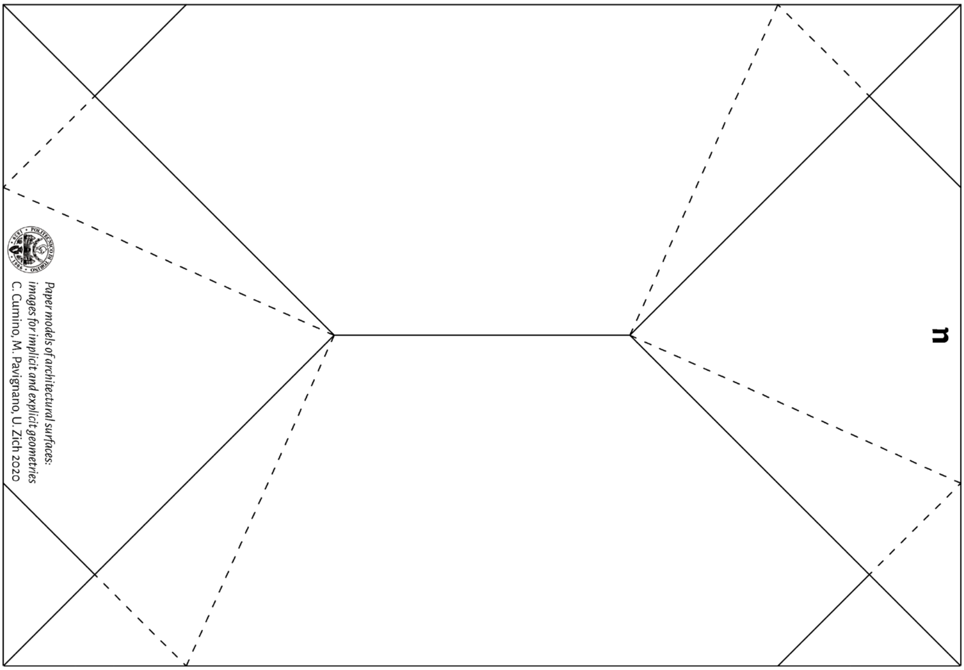
Solima, L. (2012). *Il museo in ascolto. Nuove strategie di comunicazione per i musei statali*. Soveria Manneli, IT: Rubettino.

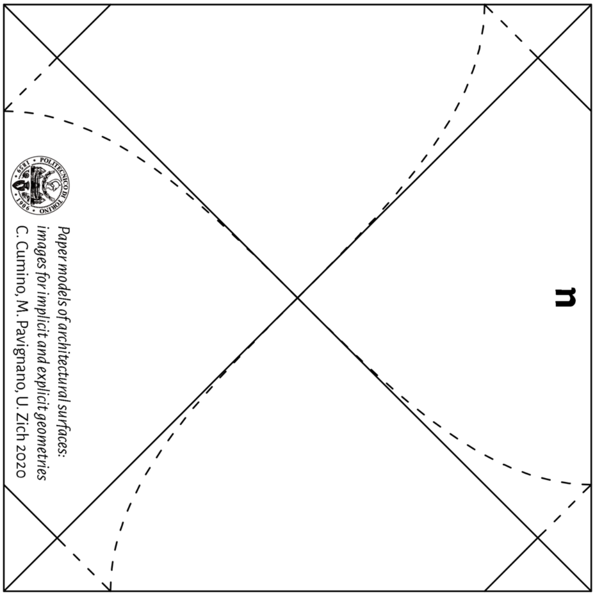
Ugo, V. (2008). *Mìmesis. Sulla critica della rappresentazione dell'architettura*. Sant'Arcangelo di Romagna, IT: Maggioli.

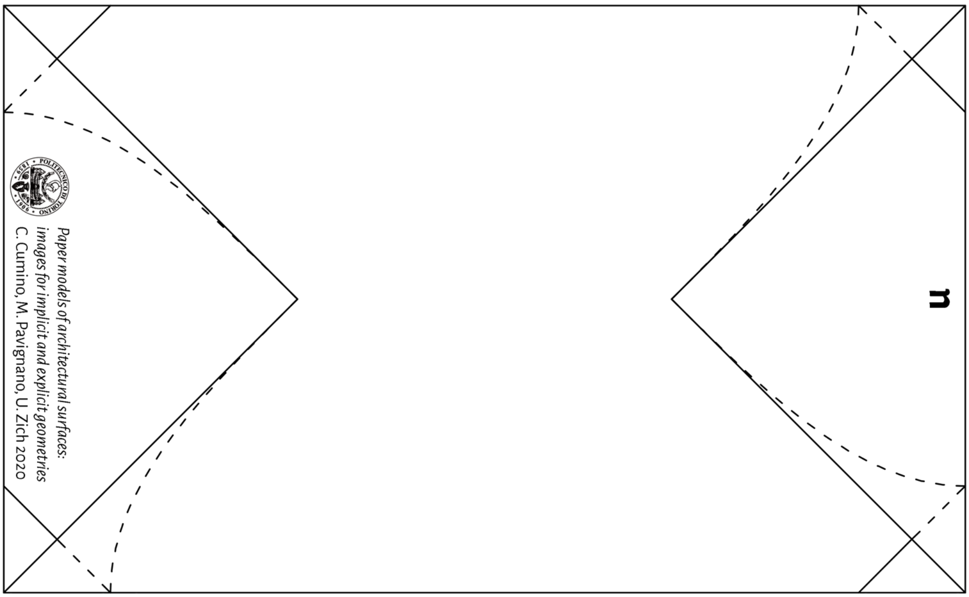
SUPPLEMENTARY MATERIALS

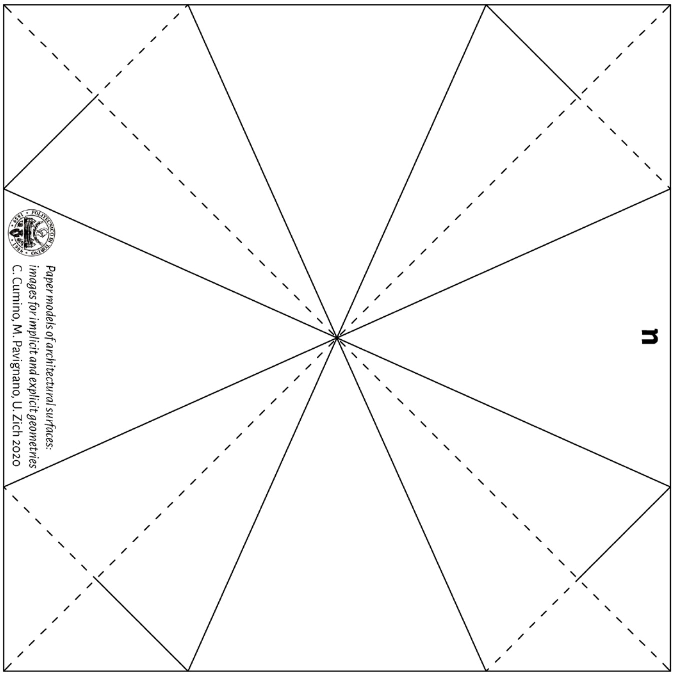
We attach the CPs of the paper models described in the paper, in order to provide the possibility for the reader to fold them and analyse our exposition with the haptic support of these tangible artefacts. The first four CPs refer to models with extra-paper outside (Table 1, A1-3, B1-3, C1-3, D1-3), the last four CPs to models with extra-paper inside (Table 1, A4-6, B4-6, C4-6, D4-6). The CPs follows the standard origami language: straight line = mountain fold, dashed line = valley fold. CPs can be printed. The 'modular unit' is the base side of pyramid and cloister vault (smaller sides for pitch roof and barrel vault with cloister heads).

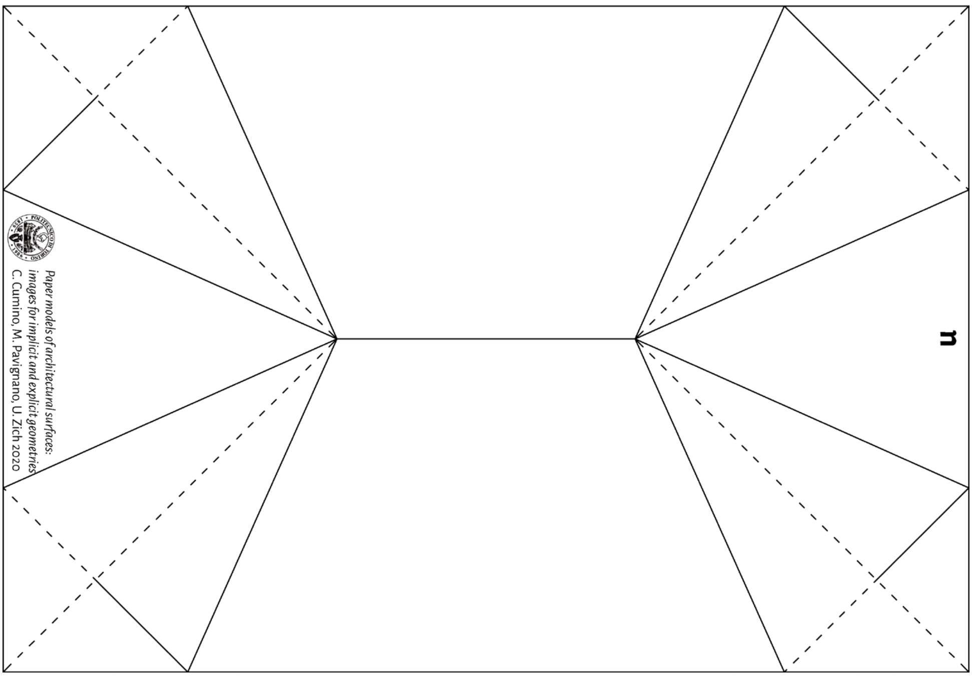

Paper models of architectural surfaces:
images for implicit and explicit geometries
C. Cumino, M. Pavignano, U. Zich 2020
n

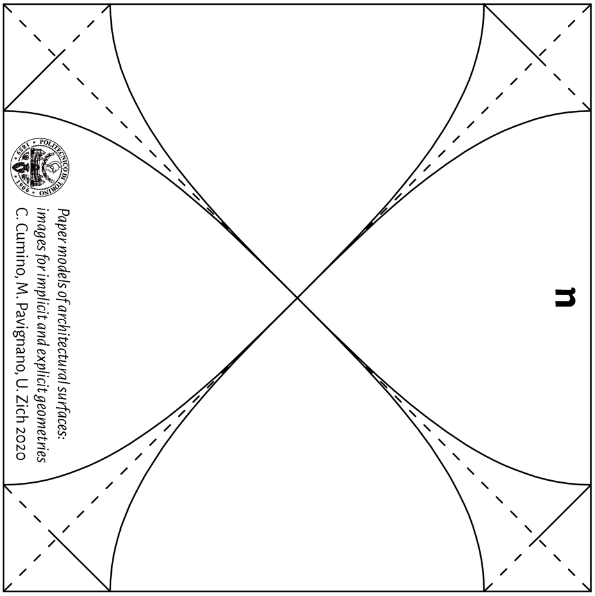

Paper models of architectural surfaces:
images for implicit and explicit geometries
C. Cumino, M. Pavignano, U. Zich 2020
n

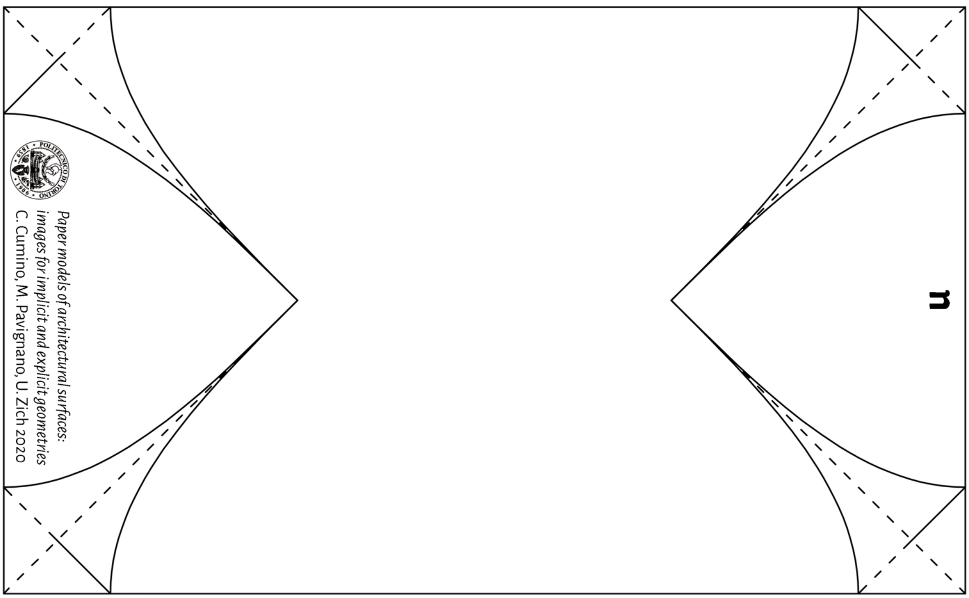

Paper models of architectural surfaces:
images for implicit and explicit geometries
C. Cumino, M. Pavignano, U. Zich 2020


Paper models of architectural surfaces:
images for implicit and explicit geometries
C. Cumino, M. Pavignano, U. Zich 2020


Paper models of architectural surfaces:
images for implicit and explicit geometries
C. Cumino, M. Pavignano, U. Zich 2020


Paper models of architectural surfaces:
images for implicit and explicit geometries
C. Cumino, M. Pavignano, U. Zich 2020


Paper models of architectural surfaces:
images for implicit and explicit geometries
C. Cumino, M. Pavignano, U. Zich 2020


Paper models of architectural surfaces:
images for implicit and explicit geometries
C. Cumino, M. Pavignano, U. Zich 2020

Article available at
DOI: 10.6092/issn.2724-2463/11116

How to cite

as article

Cumino, C., Pavignano, M., Zich, U. (2020). Paper models of architectural surfaces: images for implicit and explicit geometries. *img journal*, *2*, 86-115.

as contribution in book

Cumino, C., Pavignano, M., Zich, U. (2020). Paper models of architectural surfaces: images for implicit and explicit geometries. In E. Cicalò (Ed.), *img journal 02/2020 Graphics* (pp. 86-115). Alghero, IT: Publica. ISBN 9788899586119